Unit - 3: Fourier Transform

Fourier Integral

$$F(x) = \int_0^{\infty} (A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x)\, d\lambda$$

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(t)\cos\lambda t\, dt$$

$$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(t)\sin\lambda t\, dt$$

Fourier Cosine Integral

$$F(x) = \int_0^{\infty} A(\lambda)\cos\lambda x\, d\lambda$$

$$A(\lambda) = \frac{2}{\pi}\int_0^{\infty} F(x)\cos\lambda x\, dx$$

Fourier Sine Integral

$$F(x) = \int_0^{\infty} B(\lambda)\sin\lambda x\, d\lambda$$

$$B(\lambda) = \frac{2}{\pi}\int_0^{\infty} F(x)\sin\lambda x\, dx$$

1. Express the function $F(x) = 1$ For $|x| \le 1$
$= 0$ $\quad |x| > 1$

as a fourier Integral.

→ Sol^n. we know.

$$F(x) = \int_0^{\infty} (A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x)\, d\lambda \quad \text{—— (A)}$$

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(x)\cos\lambda x\, dx$$

$$A(\lambda) = \frac{1}{\pi}\left[\int_{-1}^{1} F(x)\cos\lambda x\, dx + \int_{1}^{\infty} F(x)\cos\lambda x\, dx\right]$$

$$= \frac{1}{\pi}\int_{-1}^{1} 1 \cos\lambda x\, dx + 0$$

$$= \frac{1}{\pi}\left[\frac{\sin\lambda x}{\lambda}\right]_{-1}^{1}$$

$$= \frac{1}{\pi\lambda}\left[\sin\lambda - \sin(-\lambda)\right]$$

$$= \frac{1}{\pi\lambda}\left[\sin\lambda + \sin\lambda\right]$$

$$= \frac{2\sin\lambda}{\pi\lambda} \quad \text{—— (B)}$$

$$B(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(x) \sin\lambda x \, dx$$

$$= \frac{1}{\pi} \left[\int_{-1}^{1} F(x) \sin\lambda x \, dx + \int_{1}^{\infty} F(x) \sin\lambda x \, dx \right]$$

$$= \frac{1}{\pi} \int_{-1}^{1} 1 \cdot \sin\lambda x \, dx + 0$$

$$= \frac{1}{\pi} \left[\frac{-\cos\lambda x}{\lambda} \right]_{-1}^{1}$$

$$= \frac{-1}{\pi\lambda} [\cos\lambda - \cos(-\lambda)]$$

$$F = \frac{-1}{\pi\lambda} [\cos\lambda - \cos\lambda]$$

$$B(\lambda) = 0 \quad \text{—— (C)}$$

From eq^n A, B & C

$$F(x) = \int_{0}^{\infty} \left(\frac{2\sin\lambda}{\pi\lambda} \cos\lambda x + 0 \sin\lambda x \right) dx$$

$$F(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin\lambda \cos\lambda x}{\lambda} \, d\lambda$$

put $x = 0$

$$f(0) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin\lambda \cos 0}{\lambda} \, d\lambda$$

$$1 = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda$$

$$\Rightarrow \int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda = \frac{\pi}{2}$$

$$\therefore \boxed{\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}}.$$

2 Show that $\int_0^\infty \frac{\omega \sin x\omega}{1+\omega^2}\, d\omega = \frac{\pi}{2} e^{-x}$ as Fourier sine transform.

→

$$F(x) = \int_0^\infty B(\lambda) \sin\lambda x \, d\lambda$$

$$B(\lambda) = \frac{2}{\pi}\int_{0}^{\infty} F(x) \sin\lambda x \, dx$$

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty e^{-x} \sin\lambda x \, dx$$

Use $\int e^{ax} \sin bx \, dx = \frac{e^{a}}{a^2+b^2}[a\sin bx - b\cos bx]$

$$a = -1,\ b = \lambda$$

$$B(\lambda) = \frac{2}{\pi}\left[\frac{e^{-x}}{(-1)^2+\lambda^2}(-\sin\lambda x - \lambda\cos\lambda x)\right]_0^\infty$$

$$= \frac{2}{\pi}\left[0 - \frac{e^0}{1+\lambda^2}(-\sin 0 - \lambda\cos 0)\right]$$

$$= \frac{2}{\pi}\left[\frac{\lambda}{1+\lambda^2}\right]$$

$$= \frac{2\lambda}{\pi(1+\lambda^2)}$$

$$F(x) = e^{-x} = \int_0^\infty \frac{2\lambda}{\pi(1+\lambda^2)} \sin\lambda x \, dx$$

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{1+\lambda^2}\, d\lambda$$

λ is replace by w

$$\boxed{\frac{\pi}{2}e^{-x} = \int_0^\infty \frac{w\sin wx}{1+w^2}\, dw}$$

3. Using cosine Fourier transform show that

$$\int_0^\infty \frac{\cos xw}{1+w^2}\, dw = \frac{\pi e^{-x}}{2}$$

→

$$F(x) = \int_0^\infty A(\lambda)\cos\lambda x \, d\lambda \quad \text{— (A)}$$

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty F(x)\cos\lambda x \, dx$$

$$= \frac{2}{\pi}\int_0^\infty e^{-x}\cos\lambda x \, dx$$

use $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$a=-1, \quad b=\lambda$

$$A(\lambda) = \frac{2}{\pi}\left[\frac{e^{-x}}{(-1)^2+\lambda^2}(-\cos\lambda x + \lambda\sin\lambda x)\right]_0^\infty$$

$$= \frac{2}{\pi}\left[0 - \frac{e^0}{1+\lambda^2}(-\cos 0 + \lambda\sin 0)\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{1+\lambda^2}\right]$$

$$A(\lambda) = \frac{2}{\pi(1+\lambda^2)}$$

eq^n (A) becomes

$$F(x) = e^{-x} = \int_0^\infty \frac{2}{\pi(1+\lambda^2)}\cos\lambda x \, d\lambda$$

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\cos\lambda x}{(1+\lambda^2)}\, d\lambda$$

λ is replaced by ω

$$\boxed{\frac{\pi}{2}e^{-x} = \int_0^\infty \frac{\cos\omega x}{1+\omega^2}\, d\omega}$$

Fourier Transform

$$F(s) = \int_{-\infty}^{\infty} F(t)\, e^{ist}\, dt$$

Inverse Fourier Transform

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

* Examples

1. Find the Fourier transform for

$F(x) = 1 - x^2 \quad |x| \le 1$ and hence evaluate

$\quad = 0 \quad |x| > 1$

$$\int_0^{\infty} \left(\frac{x\cos x - \sin x}{x^3} \right) \cos \frac{x}{2}\, dx$$

$\Longrightarrow$

We have

$F(x) = 1 - x^2 \quad |x| \le 1 \Rightarrow -1 \le x \le 1$

$\quad = 0 \quad |x| > 1 \Rightarrow 1 < x < \infty$

By using Fourier Transform

$$F(s) = \int_{-\infty}^{\infty} F(x)\, e^{isx}\, dx$$

$$= \int_{-1}^{1} F(x)(\cos sx + i\sin sx)\, dx + \int_{1}^{\infty} F(x)(\cos sx + i\sin sx)\, dx$$

$$= \int_{-1}^{1} (1 - x^2)(\cos sx + i\sin sx)\, dx + 0$$

$$F(s) = \int_{-1}^{1}(1-x^2)\cos sx\,dx + \int_{-1}^{1}(1-x^2)\, i\sin sx\,dx \quad \text{—— (A)}$$

consider

$F_1(x) = (1-x^2)\cos sx$

$F_1(-x) = (1-(-x)^2)\cos s(-x)$

$= (1-x^2)\cos sx = F_1(x)$

$\Rightarrow F_1(-x) = F_1(x)$

$\Rightarrow F_1(x)$ is even function

$F_2(x) = (1-x^2)\sin sx$

$F_2(-x) = (1-(-x)^2)\sin s(-x)$

$= (1-x^2)(-\sin sx)$

$\therefore F_2(-x) = -F(x)$

$F_2(x)$ is odd

eq^n (A) becomes

$$F(s) = \underbrace{\int_{-1}^{1}(1-x^2)\cos sx\,dx}_{\text{even function}} + \underbrace{\int_{-1}^{1}(1-x^2)\, i\sin sx\,dx}_{\text{Odd function}}$$

$$= 2\int_{0}^{1}(1-x^2)\cos sx\,dx + 0$$

$$= 2\left[(1-x^2)\frac{\sin sx}{s} - (0-2x)\frac{(-\cos sx)}{s\times s} + \frac{(-2)(-\sin sx)}{s^2\times s}\right]_0^1$$

$$= 2\left[0 - \frac{2\times 1\cos s}{s^2} + \frac{2\sin s}{s^3} - 0 - 0 - 0\right]$$

$$= 2\left[-\frac{2\cos s}{s^2}+\frac{2\sin s}{s^3}\right]$$

$$= -4\left[\frac{s\cos s-\sin s}{s^3}\right] \longrightarrow (B)$$

By using Inverse Fourier Transform

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} -4\frac{(s\cos s-\sin s)}{s^3}(\cos sx - i\sin sx)\, ds$$

$$=\frac{2(-4)}{2\pi}\int_{0}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sx\, ds$$

Here $\cos sx$ even funct^n
$\sin sx$ odd function

$$1-x^2 = F(x) = -\frac{4}{\pi}\int_0^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sx\, ds$$

put $s=x$ & $x=1/2$

$$1-(1/2)^2 = -\frac{4}{\pi}\int_0^{\infty}\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\, dx$$

$$1-\frac{1}{4} = -\frac{4}{\pi}\int_0^{\infty}\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\, dx$$

$$\frac{3}{4} = -\frac{4}{\pi}\int_0^{\infty}\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\, dx$$

$$\Rightarrow \int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx = \frac{3}{4}\times\left(\frac{-\pi}{4}\right)$$

$$\boxed{\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx = \frac{-3\pi}{16}.}$$

2. Find Fourier sine transform of $\frac{e^{-ax}}{x}$.

→ we have $F(x) = \frac{e^{-ax}}{x}$

Fourier transform of $F(x)$ is

$$Fs(s) = \int_0^\infty F(x)\sin sx\,dx$$

$$Fs(s) = \int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx = I \text{ (say)} \quad \text{—(I)}$$

diff w.r.t. s

$$\frac{dI}{ds} = \int_0^\infty \frac{e^{-ax}}{x}\cos sx \times x\,dx$$

$$\frac{dI}{ds} = \int_0^\infty e^{-ax}\cos sx\,dx$$

use. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx - b\sin bx)$

$a = -a, \quad b = s$

$$= \left[\frac{e^{-ax}}{(-a)^2+s^2}\left[-a\cos sx - s\sin sx\right]\right]_0^\infty$$

$$= 0 - \left(\frac{e^0}{a^2+s^2}(-a\cos 0 - s\sin 0)\right]$$

$$= \frac{a}{a^2+s^2}$$

$$\therefore \frac{dI}{ds} = \frac{a}{a^2+s^2}$$

$$dI = \frac{a}{a^2+s^2}\, ds$$

on integration

$$\int dI = \int \frac{a}{a^2+s^2}\, ds$$

$$I = a \times \frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right) + c$$

$$I = \tan^{-1}\left(\frac{s}{a}\right) + c \quad \text{——— (A)}$$

Put $s = 0$ in eq^n (I)

$I = F_s(s) = 0$

$\therefore I = 0$

Put this value in eq^n A

$0 = \tan^{-1}(0) + c$

$0 = 0 + c$

$\Rightarrow \boxed{c = 0}$

eq^n A becomes

$$I = F_s(s) = \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx \, dx$$

$$= \tan^{-1}\left(\frac{s}{a}\right) + c$$

$$\therefore \boxed{I = \tan^{-1}\left(\frac{s}{a}\right)}$$

Fourier Transform

$$F(s) = \int_{-\infty}^{\infty} F(t)\, e^{ist}\, dt$$

is called fourier transform

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)\, e^{isx}\, dx$$

is called inverse fourier transform

Fourier sine and cosine transform

$$F_c(s) = \int_0^{\infty} F(x) \cos sx\, dx$$

is called fourier cosine transform

$$F(x) = \frac{2}{\pi} \int_0^{\infty} F_c(s) \cos x\, ds$$

is called Inverse fourier cosine Transform

$$F_s(s) = \int_0^{\infty} F(x) \sin sx\, dx$$

is called fourier sine transform

$$F(x) = \frac{2}{\pi} \int_0^{\infty} F_s(s) \sin sx\, d\cancel{x}s$$

is called inverse fourier sine transform

1. Find the Fourier sine transform of $e^{-|x|}$ and hence show that

$$\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx = \frac{\pi e^{-m}}{2} \qquad m>0$$

→

Let $F(x) = e^{-|x|} = e^{-x}$

By using Fourier sine transform formula.

$$F_s(s) = \int_0^\infty F(x) \sin sx \, dx$$

$$F_s(s) = \int_0^\infty e^{-x} \sin sx \, dx$$

Use $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b\cos bx]$

$a=-1$, $b = s$

$$F_s(s) = \left[\frac{e^{-x}}{(-1)^2+s^2}(-\sin sx - s\cos sx)\right]_0^\infty$$

$$= 0 - \left[\frac{e^0}{1+s^2}(-\sin 0 - s\cos 0)\right]$$

$$= -\left[\frac{1}{1+s^2}(-0-s)\right]$$

$$F_s(s) = \frac{s}{1+s^2}$$

By using Inverse Fourier Sine Transform Formula

$$F(x) = \frac{2}{\pi} \int_0^\infty F_s(s) \sin sx \, ds$$

$$F(x) = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin sx \, ds$$

Put $s = x$ & $x = m$ we get

$$F(x) = e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{x \sin xm}{1+x^2} dx$$

$$\boxed{\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}}$$

2. Find the fourier cosine transform of $F(x) = \frac{1}{1+x^2}$ and hence derive the Fourier sine transform of $\phi(x) = \frac{x}{1+x^2}$

→ Sol^n, we have $F(x) = \frac{1}{1+x^2}$

$$F_c(s) = \int_0^\infty F(x) \cos sx \, dx$$

$$F_c(s) = \int_0^\infty \frac{1}{1+x^2} \cos x \, dx$$

$$I = \int_0^{\infty} \left(\frac{\cos sx}{1+x^2}\right) dx \quad \text{—— (1)}$$

diff w.r.t s

$$\frac{dI}{ds} = \int_0^{\infty} \left(\frac{1}{1+x^2}\right) (-\sin sx) x \, dx \quad \text{—— } (*)$$

$$= -\int_0^{\infty} \frac{x}{(1+x^2)} \sin sx \, dx$$

$$= -\int_0^{\infty} \frac{x^2 \sin sx}{x(1+x^2)} dx$$

$$= -\int_0^{\infty} \frac{(1+x^2-1)}{x(1+x^2)} \sin sx \, dx$$

$$= -\int_0^{\infty} \frac{1+x^2}{x(1+x^2)} \sin sx \, dx + \int_0^{\infty} \frac{1}{x(1+x^2)} \sin sx \, dx$$

$$= -\underline{\int_0^{\infty} \frac{\sin sx}{x} dx} + \int_0^{\infty} \frac{\sin sx}{x(1+x^2)} dx$$

$$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^{\infty} \frac{\sin sx}{x(1+x^2)} dx \quad \text{—— (2)}$$

Diff w r t s

$$\frac{d^2I}{ds^2} = 0 + \int_0^{\infty} \frac{\cos sx \times x}{x(1+x^2)} dx$$

$$\frac{d^2I}{ds^2} = I$$

$$\frac{d^2I}{ds^2} - I = 0$$

A.E eq^n

$m^2 - 1 = 0$

$(m-1)(m+1) = 0$

$m = 1, \quad m = -1$

$C.F = I = C_1 e^{m_1 s} + C_2 e^{m_2 s}$

$I = C_1 e^{s} + C_2 e^{-s}$ —— (III)

Diff. w.r.t. s

$$\frac{dI}{ds} = C_1 e^{s} \times 1 + C_2 e^{-s} \times (-1)$$

$$\frac{dI}{ds} = C_1 e^{s} - C_2 e^{-s} \quad \text{—— (IV)}$$

Put $s = 0$ in eq^n (I)

$$F_c(s) = \int_0^\infty \frac{1}{1+x^2} \cos 0 \, dx$$

$$F_c(s) = \int_0^\infty \frac{1}{1+x^2} \, dx$$

$$= (\tan^{-1} x)_0^\infty$$

$$I = \tan^{-1}\infty - \tan^{-1}(0)$$

$$I = \frac{\pi}{2} - 0$$

$$\boxed{I = \frac{\pi}{2}} \longrightarrow (a)$$

Put $s=0$ in eq^n (II)

$$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx$$

$$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin 0}{x(1+x^2)}\,dx$$

$$\boxed{\frac{dI}{ds} = -\frac{\pi}{2}} \quad \text{---} (b)$$

Put $s=0$ in eq^n (III)

$$\frac{\pi}{2} = C_1 e^0 + C_2 e^0 \quad \text{--- From eq}^n \text{ (a)}$$

$$\frac{\pi}{2} = C_1 + C_2 \quad \text{---} (A)$$

Put $s=0$ in eq^n (IV) and From eq^n (b)

$$-\frac{\pi}{2} = C_1 e^0 - C_2 e^0$$

$$-\frac{\pi}{2} = c_1 - c_2 \quad \text{—} \; (B)$$

$$c_1 + c_2 = \pi/2$$
$$c_1 - c_2 = -\pi/2$$
$$2c_1 + 0 = 0$$

$\boxed{c_1 = 0}$ $\boxed{c_2 = \frac{\pi}{2}}$

$$I = C.F = c_1 e^{s} + c_2 e^{-s}$$

$$= 0 e^{s} + \pi/2 \, e^{-s}$$

$$\therefore I = \frac{\pi}{2} e^{-s}$$

$$I = F_c(s) = \int_0^\infty \frac{1}{(1+x^2)} \cos sx \, dx = \frac{\pi}{2} e^{-s}$$

$$I = \frac{\pi}{2} e^{-s}$$

diff. w.r.t.s

$$\frac{dI}{ds} = \frac{\pi}{2} e^{-s} \times -1$$

$$= -\frac{\pi}{2} e^{-s}$$

From eq^n ⊛

$$\int_0^\infty \frac{-x\sin sx}{(1+x^2)}\,dx = -\frac{\pi}{2}e^{-s}$$

$$= -\int_0^\infty \frac{x}{1+x^2}\sin sx\,dx$$

$$\Rightarrow \quad -F_s(s) = -\frac{\pi}{2}e^{-s}$$

$$\boxed{F_s(s) = \int_0^\infty \frac{x}{1+x^2}\sin sx\,dx = \frac{\pi}{2}e^{-s}}$$

Finite Fourier Transform

- Finite fourier sine transform :-

$F_s(n) = \int_0^c F(x)\sin\frac{n\pi x}{c}\,dx$ is called Finite Fourier sine transform

$F(x) = \frac{2}{c}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{c}$ is called

Inverse Finite Fourier sine transform

- Finite Fourier cosine Transform

$F_c(n) = \int_0^c F(x) \cos \frac{n\pi x}{c} \, dx$ is called Finite Fourier cosine Transform.

$$F(x) = \frac{2}{c} \sum_{n=1}^{\infty} F_c(n) \frac{\cos n\pi x}{c}.$$

Examples.

1. Find Fourier cosine and sine transform of $F(x) = 2x$, $0 < x < 4$

→ Sol^n, we have

$F(x) = 2x \qquad 0 < x < 4$

$\Rightarrow \; c = 4$

Finite Fourier cosine transform

$$F_c(n) = \int_0^c F(x) \cos \frac{n\pi x}{c} \, dx$$

$$F_c(n) = \int_0^4 2x \cos \frac{n\pi x}{4} \, dx$$

$$= 2 \int_0^4 x \cos \frac{n\pi x}{4} \, dx$$

Use by parts rule : $\int uv\,dx = u\int v\,dx - \int \left[\frac{du}{dx} \int v\,dx\right]$

$$= 2\left[x\,\frac{\sin(n\pi x/4)}{n\pi/4} - 1\,\frac{1}{n\pi/4}\left(-\cos\frac{n\pi x}{4}\right)\frac{1}{n\pi/4}\right]_0^4$$

$$= 2\left[\frac{4x}{n\pi}\sin\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right]_0^4$$

$$= 2\left[\frac{4\times 4}{n\pi}\sin\frac{4n\pi}{4} + \frac{16}{n^2\pi^2}\cos\frac{4n\pi}{4} - \left(0 + \frac{16}{n^2\pi^2}\cos 0\right)\right]$$

$$= 2\left[0 + \frac{16}{n^2\pi^2}(-1)^n - \frac{16}{n^2\pi^2}\times 1\right]$$

$$= \frac{32}{n^2\pi^2}\left[(-1)^n - 1\right] = \frac{32}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

$$F_s(n) = \int_0^c F(x)\sin\frac{n\pi x}{c}\,dx$$

$$= \int_0^4 2x\sin\frac{n\pi x}{4}\,dx$$

$$= 2\int_0^4 x\sin\frac{n\pi x}{4}\,dx$$

$$= 2\left[x\,\frac{(-\cos(n\pi x/4))}{n\pi/4} - \frac{(-\sin(n\pi x/4))}{n\pi/4\times n\pi/4}\right]_0^4$$

$$= 2\left[-\frac{4x}{n\pi}\cos\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}\right]_0^4$$

$$= 2\left[\frac{-4\times 4}{n\pi}\cos\frac{4n\pi}{4} - \frac{-4\times 0\times \cos 0}{n\pi}\right]$$

$$= 2\left[\frac{-16}{n\pi}\cos n\pi\right]$$

$$= \frac{-32\cos n\pi}{n\pi}$$

Parseval's Identity:

1. $$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,G(s)\,ds = \int_{-\infty}^{\infty} F(x)\,g(x)\,dx$$

2. $$\frac{1}{2\pi}\int_{-\infty}^{\infty} (F(s))^2\,ds = \int_{-\infty}^{\infty} (F(x))^2\,dx$$

1. $$\frac{2}{\pi}\int_{0}^{\infty} F_c(s)\,G_c(s)\,ds = \int_{0}^{\infty} F(x)\,g(x)\,dx$$

2. $$\frac{2}{\pi}\int_{0}^{\infty} F_s(s)\,G_s(s)\,ds = \int_{0}^{\infty} F(x)\,g(x)\,dx$$

3. $$\frac{2}{\pi}\int_{0}^{\infty} (F_c(s))^2\,ds = \int_{0}^{\infty} (F(x))^2\,dx$$

4. $$\frac{2}{\pi}\int_{0}^{\infty} (F_s(s))^2\,ds = \int_{0}^{\infty} (F(x))^2\,dx$$

* Example :

1 Evaluate $\int_0^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)} dx$

→ Sol^n Let $F(x) = e^{-ax}$ $\quad F_c(s) = \frac{a}{a^2+s^2}$

$g(x) = e^{-bx}$ $\quad G_c(s) = \frac{b}{b^2+s^2}$

By using parseval's Identity

$$\frac{2}{\pi}\int_0^{\infty} F_c(s)\, G_c(s)\, ds = \int_0^{\infty} F(x).g(x)\, dx$$

$$\frac{2}{\pi}\int_0^{\infty}\left(\frac{a}{a^2+s^2}\cdot\frac{b}{b^2+s^2}\right)ds = \int_0^{\infty}(e^{-ax}.e^{-bx})\,dx$$

$$\frac{2ab}{\pi}\int_0^{\infty}\frac{1}{(s^2+a^2)(s^2+b^2)}\,ds = \int_0^{\infty} e^{-ax-bx}\,dx$$

$$\frac{2ab}{\pi}\int_0^{\infty}\frac{1}{(s^2+a^2)(s^2+b^2)}\,ds = \left[\frac{e^{-x(a+b)}}{-(a+b)}\right]_0^{\infty}$$

$$= 0 - \left(\frac{e^0}{-(a+b)}\right)$$

$$= \frac{1}{a+b}$$

$$\int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)} ds = \frac{\pi}{(2ab)(a+b)}$$

$$\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{(2ab)(a+b)}$$

2 Using parseval's Identity prove that

$$\int_0^\infty \frac{t^2}{(1+t^2)^2} dt$$

→

Sol^n

Let $F(x) = \frac{x}{(1+x^2)}$ then $Fs(s) = \frac{\pi}{2} e^{-s}$

By using parseval's Identity

$$\frac{2}{\pi} \int_0^\infty (Fs(s))^2 ds = \int_0^\infty (F(x))^2 ds$$

$$\frac{2}{\pi} \int_0^\infty \left(\frac{\pi}{2} e^{-s}\right)^2 ds = \int_0^\infty \left[\frac{x}{(1+x^2)}\right]^2 dx$$

$$\frac{2}{\pi} \int_0^\infty \frac{\pi^2}{4} e^{-2s} ds = \int_0^\infty \frac{x^2}{(1+x^2)^2} dx$$

$$\frac{2}{\pi} \times \frac{\pi^2}{4} \int_0^\infty e^{-2s} ds = \int_0^\infty \frac{x^2}{(1+x^2)^2} dx$$

$$\frac{\pi}{2} \left[\frac{e^{-2s}}{-2}\right]_0^\infty = \int_0^\infty \frac{x^2}{(1+x^2)^2} dx$$

$$\frac{\pi}{2} \times \frac{-1}{2}\left[0 - e^{0}\right] = \int_0^{\infty} \frac{x^2}{(1+x^2)^2}\,dx$$

$$\frac{\pi}{4} = \int_0^{\infty} \frac{x^2}{(1+x^2)^2}\,dx$$

$$\therefore \boxed{\int_0^{\infty} \frac{t^2}{(1+t^2)^2}\,dt = \frac{\pi}{4}}$$

3. By using parseval's Identity show that $\int_0^{\infty} \frac{1}{(t^2+1^2)^2}\,dt = \frac{\pi}{4}$

→

Let $F(x) = e^{-x}$ then $F_c(s) = \frac{1}{s^2+1}$

By using parseval's Identity

$$\frac{2}{\pi}\int_0^{\infty} (F_c(s))^2\,ds = \int_0^{\infty} (F(x))^2\,dx$$

$$\frac{2}{\pi}\int_0^{\infty} \left(\frac{1}{s^2+1}\right)^2 ds = \int_0^{\infty} (e^{-x})^2\,dx$$

$$\frac{2}{\pi}\int_0^{\infty} \frac{1}{(s^2+1)^2}\,ds = \int_0^{\infty} e^{-2x}\,dx$$

$$\frac{2}{\pi}\int_0^{\infty} \frac{1}{(s^2+1)^2}\,ds = \left[\frac{e^{-2x}}{-2}\right]_0^{\infty}$$

$$\frac{2}{\pi}\int_0^\infty \frac{1}{(s^2+1)^2}\, ds = -\frac{1}{2}\left[0-e^0\right]$$

$$\frac{2}{\pi}\int_0^\infty \frac{1}{(s^2+1)^2}\, ds = -\frac{1}{2}(0-1)$$

$$\int_0^\infty \frac{1}{(s^2+1)^2}\, ds = \frac{1}{2}\times\frac{\pi}{2}$$

$$\therefore \boxed{\int_0^\infty \frac{1}{(t^2+1)^2}\, dt = \frac{\pi}{4}}$$

4. By Using parseval's Identity show that $\int_0^\infty \frac{t^2}{(4+t^2)(9+t^2)}\, dt = \frac{\pi}{10}$

→ Let $F(x) = \frac{x}{4+x^2} = \frac{x}{2^2+x^2}$ then $F_s(s) = \frac{\pi}{2}e^{-2s}$

$g(x) = \frac{x}{9+x^2} = \frac{x}{3^2+x^2}$ then $G_s(s) = \frac{\pi}{2}e^{-3s}$

By Using Parseval's Identity

$$\frac{2}{\pi}\int_0^\infty F_s(s)\cdot G_s(s)\, ds = \int_0 F(x)\cdot g(x)\, dx$$

$$\cancel{\frac{2}{\pi}\int_0^\infty\left(\frac{x}{4+x^2}\cdot\frac{x}{9+x^2}\right)dx = \int_0^\infty}$$

$$\frac{2}{\pi}\int_0^\infty \left(\frac{\pi}{2}e^{-2s} \cdot \frac{\pi}{2}e^{-3s}\right) ds = \int_0^\infty \frac{x}{(4+x^2)} \frac{x}{(9+x^2)}\, dx$$

$$\frac{2}{\pi} \times \frac{\pi^2}{4}\int_0^\infty e^{-2s-3s}\, ds = \int_0^\infty \frac{x^2}{(4+x^2)(9+x^2)}\, dx$$

$$\frac{\pi}{2}\left[\frac{e^{-5s}}{-5}\right]_0^\infty = \int_0^\infty \frac{x^2}{(4+x^2)(9+x^2)}\, dx$$

$$-\frac{\pi}{10}\left[0 - e^0\right] = \int_0^\infty \frac{x^2}{(4+x^2)(9+x^2)}\, dx$$

$$-\frac{\pi}{10}\left[0-1\right] = \int_0^\infty \frac{x^2}{(4+x^2)(9+x^2)}\, dx$$

$$\frac{\pi}{10} = \int_0^\infty \frac{x^2}{(4+x^2)(9+x^2)}\, dx$$

$$\Rightarrow \boxed{\int_0^\infty \frac{t^2}{(4+t^2)(9+t^2)}\, dt = \frac{\pi}{10}}$$